PUTNAM PRACTICE SET 8

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Problem 1. Find all real numbers a with the property that the equation

$$x^2 - 2x \cdot [x] + x - a = 0$$

has two distinct nonnegative real roots.

Problem 2. Let $k, n \in \mathbb{N}$ such that $n \geq k^3 + 1$. We partition $\{1, 2, 3, \dots, 2n\}$ into k (disjoint) subsets, i.e.,

$$\{1, 2, 3, \dots, 2n\} = M_1 \cup M_2 \cup \cdots \cup M_k.$$

Prove that there exist $i, j \in \{1, \dots, k\}$ (possibly $i = j$) and there exist $(k + 1)$ distinct numbers

$$x_1, \dots, x_{k+1} \in \{1, \dots, n\}$$

such that $2x_1, \dots, 2x_{k+1} \in M_i$ and $2x_1 - 1, \dots, 2x_{k+1} - 1 \in M_j$.

Problem 3. In a finite sequence $\{x_n\}_{1 \leq n \leq m}$ of integers, the sum of each consecutive 5 numbers in the sequence is negative, while the sum of each 7 consecutive numbers in the sequence is positive. Find with proof the largest value for m.

Problem 4. Let $n \in \mathbb{N}$ and let S be the set of all tuples $(a_1, \dots, a_n)$ satisfying $a_i \in \{-1, 1\}$ for each $i = 1, \dots, n$. For two elements $x, y \in S$ of the form $x := (a_1, \dots, a_n)$ and $y := (b_1, \dots, b_n)$, we define

$$x \cdot y := (a_1 b_1, a_2 b_2, \cdots, a_n b_n) \in S.$$

Let $B \subseteq S$ be a subset with $k \geq 1$ elements. Prove that there exists some $x_0 \in S$ such that the subset of S given by

$$x_0 \cdot B := \{x_0 \cdot y \colon y \in B\}$$

intersects B in a set with at most $\frac{k^2}{2^n}$ elements.